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MATRIX THEORY

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Let $A_n \langle z \rangle$ be the n-square $(0,1)$-circulant of the type

$$\langle z \rangle = (0, t_2, t_3, \ldots, t_{k-1}, t_k).$$

Define the *associated polynomial* of type $\langle z \rangle$:

$$\lambda^t - \lambda^{t-t_2} - \lambda^{t-t_3} - \cdots - \lambda^{t-t_{k-1}} - 1,$$

where $t = t_k$. The key matrices in the construction of our recurrence relations are: Π_1, the companion matrix of the associated polynomial, and $\hat{\Pi}_r$, its rth permanental compound, $r = 1, 2, \ldots, t - 1$ (that is, Π_r is the $\binom{n}{r}$-square matrix whose entries are subpermanents of order r of Π_1, arranged in lexicographic order in rows and columns). Let the product of the distinct polynomials of these permanental compounds be

$$f(\lambda) = \prod_{r=1}^{t-1} \det(\lambda I - \Pi_r) = \lambda^m - \sum_{i=1}^{m} c_i \lambda^{m-i}. \qquad (3)$$

It is shown in [1] that $f(\lambda)$ induces the recurrence relation

$$\text{per}(A_n \langle z \rangle) = \sum_{i=1}^{m} c_i \, \text{per}(A_{n-i} \langle z \rangle) + 2f(1). \qquad (4)$$

Clearly the method will yield a recurrence formula for permanents of $(0,1)$-circulants of any type, provided that the characteristic polynomials of matrices Π_r, $r = 1, 2, \ldots, t - 1$, can be computed. Now, Π_r is a $\binom{t}{r}$-square matrix, and therefore the largest of its permanental compounds is $\binom{t}{[t/2]}$-square.

Its characteristic polynomial can be computed if t is not excessively large. Even if t is as large as 12, it should be possible to compute the 6th permanental compound of the 12-square companion matrix of the associated polynomial, if not directly then by constructing it as a transformation matrix (see [1]), and hopefully the characteristic polynomial of such 924-square matrix can be evaluated. Obviously there are many $(0,1)$-circulant types with $t \leq 12$, and for each of them our method can be used to produce a linear recurrence relation. Unfortunately polynomial (3) cannot be computed by direct methods for larger t. The problem is how to find an efficient algorithm

for constructing the characteristic polynomial of the rth permanental compound of a matrix M directly from the characteristic polynomial of M (see [2]). The problem is unsolved.

Even if a recurrence formula can be constructed for a certain type of $(0,1)$-circulants, it cannot be used for the purpose of evaluation of permanents until the appropriate initial values are computed, and these cannot be evaluated, by any known method, for $t \geq 5$. For larger values of t, the most that can be hoped for is the evaluation of the asymptotic function

$$\theta \langle z \rangle = \lim_{n \to \omega} (\mathrm{per}(A_n \langle z \rangle))^{1/n}.$$

The linear recurrence relation (4) implies that

$$\mathrm{per}(A_n \langle z \rangle) = \sum_{i=1}^{m} d_i \rho_i^n + d_{m+1},$$

where the d_i are constants, and the ρ_i are the roots of polynomial $f(\lambda)$ in (3), $\rho_1 \geq |\rho_2| \geq \cdots \geq |\rho_m|$. Hence

$$\theta \langle z \rangle = \rho_1, \tag{5}$$

where ρ_1 is the largest root of polynomial $f(\lambda)$, and thus the largest of the Perron roots of the permanental compounds of Π_1. Unfortunately, it is not known how to evaluate the Perron root of a permanental compound of a given nonnegative matrix, even if the latter happens to be a companion $(0,1)$-matrix, without actually constructing the permanental compound, which is feasible for small t only (see [2]). Nevertheless, we can obtain an upper bound for $\theta \langle z \rangle$ in terms of Perron roots of Π_1 and Π_{t-1}, which are $t \times t$ matrices, in the following way.

Minc showed in [3] that:

THEOREM I. *If Π_1 is the companion matrix of the polynomial*

$$\lambda^t - a_1 \lambda^{t-1} - a_2 \lambda^{t-2} - \cdots - a_{t-2} \lambda^2 - a_{t-1} \lambda - 1,$$

then Π_{t-1}, the $(t-1)$st permanental compound of Π_1, is the transpose of the companion matrix of the polynomial

$$\lambda^t - a_{t-1} \lambda^{t-1} - a_{t-2} \lambda^{t-2} - \cdots - a_2 \lambda^2 - a_1 \lambda - 1.$$

THEOREM II. *The rth permanental compound of Π_1 is permutationally similar to the $(t-r)$th permanental compound of Π_{t-1}, $2 \leq r \leq t-2$.*

The rth permanental compound of a matrix A is a principal submatrix of $P_r(A)$, the rth induced matrix of A. Thus if A is nonnegative then the Perron root of the rth permanental compound matrix of A cannot exceed that of $P_r(A)$. Recall that if α is the Perron root of A then the Perron root of $P_r(A)$, its induced matrix is α^r. Now, for any type of $(0,1)$-circulant, matrices Π_1 and Π_{t-1} can be written out without any difficulty (see Theorem I), and their Perron roots can be computed by standard methods. Let p and q be the Perron roots of Π_1 and Π_{t-1}, respectively. Then the Perron roots of $P_i(\Pi_1)$ and

$P_{t-i}(\Pi_{t-1})$ are p^i and q^{t-i}. It follows from equality (5), Theorem II, and the above remarks that

$$\theta\langle z\rangle \leq \max_i \min \{p^i, q^{t-i}\}.$$

Unlike the problem of finding the Perron root of a permanental compound, the corresponding determinantal problem does not present serious difficulties: the largest roots in modulus of the rth (determinantal) compound of any matrix are just the products of the r largest (in modulus) eigenvalues of the matrix. In [3] we attempt to transform our permanental problem into a more tractable determinantal problem.

Let $|M|$ denote the matrix obtained from matrix M by replacing each entry of M by its absolute value. A real $t \times t$ matrix A is said to be *r-convertible* if there exists a matrix B such that (i) $|B| = |A|$, and (ii) the rth (determinantal) compound of B is equal to the rth permanental compound of A. Matrix A is called *convertible* if it is r-convertible for every r, $2 \leq r \leq t - 1$. Question: Are the companion matrices of polynomials associated with $(0,1)$-circulants convertible? It is shown in [3] that the only nontrivial $(0,1)$-circulants whose associated matrices have convertible companion matrices are of the following three forms:

$$I + P + P^{t-1} + P^t, \text{ or } I + P + P^t, \text{ or } I + P^{t-1} + P^t.$$

The answer is rather disappointing, although the three forms represent a variety of interesting types.

Let $\lambda_1, \lambda_2, \ldots, \lambda_t$, where $\lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_t|$, and $\mu_1, \mu_2, \ldots, \mu_t$, where $|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_t|$, be the roots of $\lambda^t - \lambda^{t-1} - \lambda - 1$ and $\lambda^t - \lambda^{t-1} + \lambda + 1$, respectively. The following result, obtained in [3], is established by *converting* the companion matrix of the first of these polynomials into matrix B which satisfies conditions (i) and (ii) above.

THEOREM III. *Let*

$$A_n \langle z \rangle = I_n + P + P^{t-1} + P^t.$$

If $t \equiv 0 \mod 4$, *and* $t = 2s$, *then*

$$\theta \langle z \rangle = \max \{\lambda_1 \lambda_2 \cdots \lambda_{s-1}, \quad \mu_1 \mu_2 \cdots \mu_s\}.$$

If $t \equiv 2 \mod 4$, *and* $t = 2s$, *then*

$$\theta \langle z \rangle = \max \{\lambda_1 \lambda_2 \cdots \lambda_s, \quad \mu_1 \mu_2 \cdots \mu_{s-1}\}.$$

If t *is odd, then*

$$\theta \langle z \rangle = \max_i \{\lambda_1 \lambda_2 \cdots \lambda_{2i-1}\}.$$

Now let $\lambda_1', \lambda_2', \ldots, \lambda_t'$, where $\lambda_1' > |\lambda_2'| \geq |\lambda_3'| \geq \cdots \geq |\lambda_t'|$, and $\mu_1', \mu_2', \ldots, \mu_t'$, where $|\mu_1'| \geq |\mu_2'| \geq \cdots \geq |\mu_t'|$, be the roots of $\lambda^t - \lambda^{t-1} - 1$ and $\lambda^t - \lambda^{t-1} + 1$, respectively. We have the following result [3].

THEOREM IV.

Let $A_n \langle z' \rangle = I_n + P + P^t$ *and* $A_n \langle z'' \rangle = I_n + P^{t-1} + P^t$. *Then*

$$\theta \langle z' \rangle = \theta \langle z'' \rangle = \max \left\{ \max_i \{\lambda_1' \lambda_2' \cdots \lambda_{2i-1}'\}, \quad \max_j \{\mu_1' \mu_2' \cdots \mu_{2j}'\} \right\}.$$

The products of roots in Theorems III and IV can be conveniently evaluated by means of the classical root-squaring method (see [3]).

The results obtained in papers [1] and [3] were presented by Minc at the 1986 International Congress of Mathematicians in Berkeley.

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2. MINIMUM PERMANENTS OF DOUBLY STOCHASTIC MATRICES WITH PRESCRIBED ZERO ENTRIES.

A doubly stochastic matrix is a nonnegative matrix all of whose row and column sums are 1. The set of $n \times n$ doubly stochastic matrices is designated by Ω_n. The $n \times n$ doubly stochastic matrix all of whose entries are $1/n$ will be denoted by J_n.

In 1916 König showed that the permanent of any doubly stochastic matrix is positive. This gave rise to the famed van der Waerden permanent conjecture:

If $A \in \Omega_n$ and $A \neq J_n$, then $\operatorname{per}(A) > \operatorname{per}(J_n) = n!/n^n$.

In other words: *The permanent function achieves its minimum in Ω_n uniquely at J_n.*

In 1959 Marcus and Newman proved the conjecture holds for $n = 3$. In 1968 Eberlein and Mudholkar proved it for $n = 4$, and a year later Eberlein proved the conjecture for $n = 5$. Marcus and Newman showed that the van der Waerden conjecture holds for positive semidefinite doubly stochastic matrices, and this result was later extended by others to larger classes of doubly stochastic matrices. In 1962 Marcus and Minc proved that the permanent of any matrix in Ω_n is larger than $1/n^n$, and in 1979 Friedland improved this bound to $1/e^n$. The story reached its climax in 1981 when Egoryčev and Falikman independently proved the van der Waerden conjecture. The main idea of the proof is to show that all permanental cofactors of a *minimizing* matrix in Ω_n (i.e., of a matrix with minimum permanent in Ω_n) are equal to the permanent of the matrix. This implies that if any columns (or rows) of a minimizing matrix are replaced by their average, then the resulting matrix is also minimizing in Ω_n. This *averaging process* can be also used to show that J_n is the only minimizing matrix in Ω_n.

After the appearance of Egoryčev's and Falikman's proofs many efforts have been made to exploit their techniques in problems of determination of the minimum permanents in various faces of Ω_n. A face of Ω_n can be defined by specifying the position of prescribed fixed zeros. Specifically, a subset Z of $N \times N$, where $N = \{1, 2, \ldots, n\}$, defines the face

$\Omega_n(Z) = \{S = (s_{ij}) \in \Omega_n | s_{ij} = 0 \text{ whenever } (i,j) \in Z\}$,

provided that the set is not empty. Unfortunately the key technique of averaging lines of a minimizing matrix in a face $\Omega_n(Z)$ is severely restricted by the presence of zeros in fixed positions. In fact, this technique cannot be used in some cases at all; e.g., in case of a face consisting of matrices with zero

main diagonal. On the other hand, it can be shown that theorems of Marcus and Newman and of London on minimizing matrices in Ω_n can be extended to minimizing matrices in a face of Ω_n:

(i)   *If $A = (a_{ij})$ is a minimizing matrix in $\Omega_n(Z)$ and $a_{st} > 0$, then $per(A(s|t)) = per(A)$.*

(ii)   *If $A = (a_{ij})$ is a minimizing matrix in $\Omega_n(Z)$ and $a_{st} \in Z$, then $per(A(s|t)) \geq per(A)$*

Knopp and Sinkhorn determined the minimum permanent in a face of Ω_n with exactly one prescribed zero, and Friedland extended this result to faces in which prescribed zeros form a submatrix. Both results can be easily obtained by means of the averaging process. In 1984 Minc found the minimum permanents in all faces $\Omega_n(Z)$ in which Z is restricted to two rows or two columns. This is a complicated result. In fact, the minimum is, in general, an irrational number.

The problem of determining the minimum permanent for doubly stochastic $n \times n$ matrices with k prescribed zeros on their main diagonal (or any other diagonal) presents considerable difficulties , since the Egorycév-Falikman techniques can have only a limited application.   For $k \geq 3$ the problem is completely unsolved, except for the case $k = n = 4$ which was solved recently by London and Minc [7] who showed that the minimum permanent in this case is equal to 1/9, and it occurs only for the matrix all of whose off-diagonal entries are 1/3. It is hoped that this result will allow us to solve the case $k = 3$ and $n = 4$, which is still unsolved. In fact, apart from the London-Minc result, it is not known, for $k \geq 3$ and $n \geq 4$, whether in the set of $n \times n$ doubly stochastic matrices with k prescribed zeros in the main diagonal the permanent function must achieve its minimum at a symmetric matrix.   In 1985 Brualdi conjectured that for $k = n - 1$ the minimum is achieved uniquely for the matrix whose last row and last column entries are all equal to 1/n, and all its other off-diagonal entries are $(n-1)/n(n-2)$.   In [5] Minc showed that Brualdi's conjecture is false for all $n \geq 5$, and determined the unique minimizing matrix assuming that in the set of doubly stochastic $n \times n$ matrices with zeros in the first $k - 1$ main diagonal positions, the permanent is minimum for a matrix whose off-diagonal entries ouside its last row and last column are all equal (the assumption is tantamount to an affirmative answer to a problem proposed by Brualdi). The general problem appear to be very difficult. It is unsolved even for $n = 4$. In this case, of course, $k = 3$.   The general problem for $k = 3$ is also unsolved. In [8] Minc has found the minimum permanent for $k = 3$ and for *any* n, assuming that the minimum does occur at a symmetric matrix (see Theorem V below).

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The proof of the London-Minc result amounts to finding the minimum of the permanent of matrices of the form

$$\begin{bmatrix} 0 & \alpha-\epsilon_1 & \beta+\epsilon_2 & \gamma+\epsilon_1-\epsilon_2 \\ \alpha+\epsilon_1 & 0 & \gamma-\epsilon_3 & \beta-\epsilon_1+\epsilon_3 \\ \beta-\epsilon_2 & \gamma+\epsilon_3 & 0 & \alpha+\epsilon_2-\epsilon_3 \\ \gamma-\epsilon_1+\epsilon_2 & \beta+\epsilon_1-\epsilon_3 & \alpha-\epsilon_2+\epsilon_3 & 0 \end{bmatrix},$$

where α, β, γ are nonnnegative numbers satisfying

$$\alpha + \beta + \gamma = 1,$$

and ϵ_1, ϵ_2, ϵ_3 satisfy

$$\begin{cases} |\epsilon_2 - \epsilon_3|, & |\epsilon_1| \le \alpha, \\ |\epsilon_3 - \epsilon_1|, & |\epsilon_2| \le \beta, \\ |\epsilon_1 - \epsilon_2|, & |\epsilon_3| \le \gamma. \end{cases}$$

Expanding the permanent of the above matrix we obtain, after some simplifications,

$$\begin{aligned} \text{per}(A) = {} & (\alpha^2 + \beta^2 + \gamma^2)^2 + (2\alpha\beta - \gamma^2)(\epsilon_1 + \epsilon_2 - \epsilon_3)^2 \\ & + (2\alpha\gamma - \beta^2)(\epsilon_1 - \epsilon_2 + \epsilon_3)^2 + (2\beta\gamma - \alpha^2)(-\epsilon_1 + \epsilon_2 + \epsilon_3)^2 \\ & + \epsilon_1^2(\epsilon_2 - \epsilon_3)^2 + \epsilon_2^2(\epsilon_1 - \epsilon_3)^2 + \epsilon_3^2(\epsilon_1 - \epsilon_2)^2 \\ & + 2\epsilon_1\epsilon_2(\epsilon_3 - \epsilon_1)(\epsilon_3 - \epsilon_2) + 2\epsilon_1\epsilon_3(\epsilon_2 - \epsilon_1)(\epsilon_2 - \epsilon_3) \\ & + 2\epsilon_2\epsilon_3(\epsilon_1 - \epsilon_2)(\epsilon_1 - \epsilon_3). \end{aligned}$$

In order to have a more manageable expression for $\text{per}(A)$, we apply the transformation

$$\begin{cases} x = \epsilon_1 + \epsilon_2 - \epsilon_3, \\ y = \epsilon_1 - \epsilon_2 + \epsilon_3, \\ z = -\epsilon_1 + \epsilon_2 + \epsilon_3, \end{cases}$$

We can then reduce our theorem to the following optimization problem:

Given the function

$$P = P(\alpha,\beta,\gamma,x,y,z)$$
$$= (\alpha^2 + \beta^2 + \gamma^2)^2 + (2\alpha\beta - \gamma^2)x^2 + (2\alpha\gamma - \beta^2)y^2 + (2\beta\gamma - \alpha^2)z^2$$
$$+ \frac{1}{8}\left[(x^2 - y^2)^2 + (y^2 - z^2)^2 + (z^2 - x^2)^2\right],$$

prove that

$$\min P(\alpha,\beta,\gamma,x,y,z) = 1/9,$$

where the minimum is over all $(\alpha,\beta,\gamma,x,y,z)$ *satisfying the constraints*

$$\alpha, \ \beta, \ \gamma \geq 0, \quad \alpha + \beta + \gamma = 1,$$

$$x, \ y, \ z \geq 0, \quad x + y \leq 2\alpha, \quad x + z \leq 2\beta, \quad y + z \leq 2\gamma,$$

and that the minimum is attained uniquely for $\alpha = \beta = \gamma = 1/3$, $x = y = z = 0$.

The problem is solved in [7].

In [8] the following result is obtained.

THEOREM V. *If a minimizing matrix in* $\Omega_n(Z)$, *where*
$Z = \{(1, 1), (2, 2), (3, 3)\}$, *is symmetric, then its permanent is equal to the permanent of*

$$\begin{bmatrix}
0 & \gamma & \gamma & \alpha & \alpha & \cdots & \alpha \\
\gamma & 0 & \gamma & \alpha & \alpha & \cdots & \alpha \\
\gamma & \gamma & 0 & \alpha & \alpha & \cdots & \alpha \\
\alpha & \alpha & \alpha & \beta & \beta & \cdots & \beta \\
\alpha & \alpha & \alpha & \beta & \beta & \cdots & \beta \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\alpha & \alpha & \alpha & \beta & \beta & \cdots & \beta
\end{bmatrix},$$

where α, β, γ *are positive numbers satisfying*

$$2\gamma + (n-3)\alpha = 3\alpha + (n-3)\beta = 1, \quad and$$

$$\alpha(3\beta^2\gamma^2 + 4(n-4)\alpha^2\beta\gamma + (n-4)(n-5)\alpha^4)$$

$$= \beta(\beta^2\gamma^2 + 3(n-3)\alpha^2\beta\gamma + (n-3)(n-4)\alpha^4).$$

The minimum permanent is then equal to

$$(n-3)!\beta^{n-5}(\beta^2\gamma^2 + 3(n-3)\alpha^2\beta\gamma + (n-3)(n-4)\alpha^4)$$

$$= (n-3)!\beta^{n-5}(c_4\beta^4 + c_3\beta^3 + c_2\beta^2 + c_1\beta + c_0),$$

where

$$c_4 = (n-3)^4(4n^2 - 10n + 3), \quad c_3 = -2(n-3)^2(8n^3 - 53n^2 + 84n - 18),$$

$$c_2 = 3(8n^4 - 86n^3 + 309n^2 - 414n + 162),$$

$$c_1 = -2(n-3)(8n^2 - 47n + 42), \quad and \quad c_0 = 4(n-3)(n-4).$$

For $n = 5$, *the permanent is minimum for the matrix of form* (1)
with $\alpha = 0.2214294$, $\beta = 0.1678559$, $\gamma = 0.2785708$ *(correct to
seven significant figures; all three values being irrational).
The minimum permanent is equal to* 0.04150119.

For $n = 4$, *the permanent is minimum for the matrix of the
form* (1) *with* $\alpha = \beta = 1/4$, $\gamma = 3/8$. *The minimum permanent is
equal to* $27/256$.

*There is only one symmetric doubly stochastic matrix with
three zeros in a diagonal; its permanent is* $1/4$.

The main tool in proving the theorem is the following lemma.

Let A be a minimizing matrix in a face $\Omega_n(Z)$. If rows $j_1, j_2, \ldots, j_t$ of A have the same Z pattern, then the matrix obtained from A by replacing each of these rows by their average is also minimizing in $\Omega_n(Z)$. An analogous result holds for columns of A with the same Z pattern.

The lemma implies that if there exists a symmetric minimizing matrix in $\Omega_n\langle z\rangle$, then the face $\Omega_n\langle z\rangle$ contains also a minimizing matrix of the form

$$
\begin{bmatrix}
0 & \gamma_1 & \gamma_2 & \alpha_3 & \alpha_3 & \cdots & \alpha_3 \\
\gamma_1 & 0 & \gamma_3 & \alpha_2 & \alpha_2 & \cdots & \alpha_2 \\
\gamma_2 & \gamma_3 & 0 & \alpha_1 & \alpha_1 & \cdots & \alpha_1 \\
\alpha_3 & \alpha_2 & \alpha_1 & \beta & \beta & \cdots & \beta \\
\alpha_3 & \alpha_2 & \alpha_1 & \beta & \beta & \cdots & \beta \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\alpha_3 & \alpha_2 & \alpha_1 & \beta & \beta & \cdots & \beta
\end{bmatrix} . \qquad (6)
$$

Now, if all the entries in the minimizing matrix (6), other than the three leading entries in the main diagonal, are positive, then it can be shown, by repeated use of theorem (i) on page 7, that $\alpha_1 = \alpha_2 = \alpha_3$. The minimum value of the permanent can be then computed. Finally it is proved in [8], using theorems (i) and (ii) on page 7, that if any of the α_i or the γ_i is zero, or if β is zero, then the matrix of form (6) cannot be minimizing in $\Omega_n(Z)$.

The results obtained in papers [5], [7], [8] together with parts of section 5.5 in book [6] were presented by Minc in a plenary address at the 1988 St. Andrews Mathematical Colloquium of the Edinburgh Mathematical Society.

3. NONNEGATIVE MATRICES.

The monograph *Nonnegative Matrices* [6] is an advanced book on all aspect of the theory of nonnegative matrices and some of its applications. It contains seven chapters: on spectral properties of nonnegative matrices, localization of the maximal eigenvalue, primitive and imprimitive matrices, structural properties of nonnegative matrices, doubly stochastic matrices (including a self-contained proof of the van der Waerden conjecture), other classes of nonnegative matrices (stochastic matrices, totally nonnegative matrices, oscillatory matrices and M-matrices), and on inverse eigenvalue problems for nonnegative matrices. The work explores some of the most recent developments in the theory of nonnegative matrices. The last two chapters of the book were written during the period of the contract.

Henryk Minc

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   three prescribed zero entries (submitted for publication).

linear recurrence relations for permanents of (0,1)-circulants of type (0, 1, 2,..., k-1). In [1] Minc generalized their method to *any* (0,1)-circulants.

1

method can be used to produce a linear recurrence relation.
Unfortunately polynomial (3) cannot be computed by direct methods
for larger t. The problem is how to find an efficient algorithm

THEOREM II. *The rth permanental compound of* Π_1 *is permutationally similar to the* $(t - r)th$ *permanental compound of* Π_{t-1}, $2 \leq r \leq t - 2$.

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into matrix B which satisfies conditions (1) and (11) above.

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